Comonadic Semantics for Description Logic Games

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Abstract

A categorical approach to study model comparison games in terms of comonads was recently initiated by Abramsky et al. In this work, we analyse games that appear naturally in the context of description logics and supplement them with suitable game comonads. More precisely, we consider expressive sublogics of $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$, namely, the logics that extend $\mathcal{ALC}$ with any combination of inverses, nominals, safe boolean roles combinations and the Self operator. Our construction augments and modifies the so-called modal comonad by Abramsky and Shah. The approach that we took heavily relies on the use of relative comonads, which we leverage to encapsulate additional capabilities within the bisimulation games in a compositional manner.

Keywords

comonads, category theory, bisimulations, expressive power, games, coalgebraic semantics

1. Introduction

Following [1], there are two different views on the fundamental features of computation, that can be summarised as "structure" and "power" as follows:

- **Structure**: Compositionality and semantics, addressing the question of mastering the complexity of computer systems and software.
- **Power**: Expressiveness and complexity, addressing the question of how we can harness the power of computation and recognize its limits.

It turned out that there are almost disjoint communities of researchers studying Structure and Power, with seemingly no common technical language and tools. To encounter this issue, Samson Abramsky and Anuj Dawar started a project, whose goal is to provide category-theoretical toolkit to reason about finite model theory. Their approach, described *e.g.* in [2], employs comonads on the category of relational structures in order to capture model comparison games such as Ehrenfeucht-Fraïssé, pebbling, and bisimulation games [2] as well as games for Hybrid logics [3] and Guarded Fragment [4]. The structure allows us to leverage the tool of category theory, and apply it to generalise known established theorems, as it was done in [5, 6].

In this paper, we continue the exploration of suitable game comonads by incorporating the comonadic semantics for description logics games, namely, for expressive description logics

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between $\mathcal{ALC}$ and $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$.[1] It is also worth mentioning a parallel research that defines categorical semantics for $\mathcal{ALC}$ [7, 8], however, the approach is much different from ours, as we focus solely on games and leave $\mathcal{ALC}$ in the standard set-theoretic semantics.

1.1. Our Results

In what follows, we change setting established in the previous work from the category of relational structures, to a category of pointed interpretations that are parametrised by subsets of role names, concept names and individual names.

We start with defining comonadic semantics for $\mathcal{ALC}$-bisimulation-games. It is well-known that $\mathcal{ALC}$ is a notational variant of a multi-modal logic; hence, we employ this observation to take the full advantage of existing results on modal logic from [2] and use them as the base for our further investigations. In order to define comonadic semantics for DLs $\mathcal{L} \subseteq \mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$, instead of providing it directly for them (and thus repeating all the required proofs from [2]), we follow a different route. We provide a family of game reductions from $\mathcal{L}$ to weaker sublogics, ending up on $\mathcal{ALC}$, which transform interpretations in such a way that a winning strategy in $\mathcal{L}$-bisimulation-game is equivalent to a winning strategy in $\mathcal{ALC}$-bisimulation-game for a suitably transformed interpretations. From a categorical point of view, we introduce a comonad for $\mathcal{ALC}$ logic and reductions shall be defined by functors, on which we will build relative comonads to encapsulate the additional capabilities available in an $\mathcal{L}$-bisimulation-game. By composing the reduction functors together, we shall obtain comonadic semantics for all of the games for considered logics.

2. Preliminaries

We start with a recap of notions from category theory [9, 10], such as comonads, as well as from description logics, for which we define their syntax, semantics and bisimulations [11]. By doing so, we would like to unify the context for readers from different backgrounds.

2.1. Preliminaries on DLs

We fix infinite mutually disjoint sets of *individual names* $\mathbf{N_I}$, *concept names* $\mathbf{N_C}$, and *role names* $\mathbf{N_R}$. We will briefly recap syntax and semantics of $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$-concepts and as well as $\mathcal{L}$-concepts for relevant sublogics $\mathcal{L}$ of $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$. The following EBNF grammar defines *atomic concepts* B, *concepts* C, *atomic roles* r, *simple roles* s with $\mathrm{o} \in \mathbf{N_I}$, $\mathrm{A} \in \mathbf{N_C}$, $p \in \mathbf{N_R}$:

$$\begin{array}{rclrcl} \mathrm{B} & ::= & \mathrm{A} \mid \{\mathrm{o}\}, & \mathrm{C} & ::= & \mathrm{B} \mid \neg\mathrm{C} \mid \mathrm{C} \sqcap \mathrm{C} \mid \exists s.\mathrm{C} \mid \exists s.\mathsf{Self} \\ r & ::= & p \mid p^{-}, & s & ::= & r \mid s \cap s \mid s \cup s \mid s \setminus s \end{array}$$

The semantics of $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$-concepts is defined via *interpretations* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ composed of a non-empty set $\Delta^{\mathcal{I}}$ called the *domain of* $\mathcal{I}$ and an *interpretation function* $\cdot^{\mathcal{I}}$ mapping individual names to elements of $\Delta^{\mathcal{I}}$, concept names to subsets of $\Delta^{\mathcal{I}}$, and role names to subsets of

[1]It will become clear why we write $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$ instead of $\mathcal{ALCOI}b_{\mathsf{Self}}$ later.

$\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. This mapping is then extended to complex concepts and roles (*cf.* Table 1). The *rank* of a concept is the maximal nesting-depth of $\exists$-restrictions.

The $\mathcal{ALC}$-concepts are obtained by dropping from the syntax the inversions of roles ($\mathcal{I}$), safe boolean combination of roles (b) (*i.e.* union, intersection and difference), nominals ($\mathcal{O}$) and the Self operator (Self). $\mathcal{L}$-concepts for other sublogics $\mathcal{L}$ of $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$ are defined similarly. We stress here that role union/intersection/difference, the Self operator, role inverse $\cdot^{-}$ and nominals $\{\cdot\}$ are operators and they do not introduce neither new role names nor new concept names. We will find it convenient to employ expressions of the form $\mathcal{ALC}\Phi$ or $\mathcal{L}\Phi$ with $\Phi \subseteq \{\mathcal{O}, \mathcal{I}, \mathsf{Self}, b\}$ to speak collectively about different expressive sublogics of $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$.

Name	Syntax	Semantics
conc. negation	$\neg\mathrm{C}$	$\Delta^{\mathcal{I}} \setminus \mathrm{C}^{\mathcal{I}}$
conc. intersection	$\mathrm{C} \sqcap \mathrm{D}$	$\mathrm{C}^{\mathcal{I}} \cap \mathrm{D}^{\mathcal{I}}$
exist. restriction	$\exists r.\mathrm{C}$	$\{\, \mathrm{d} \mid \exists \mathrm{e}.(\mathrm{d}, \mathrm{e}) \in r^{\mathcal{I}} \wedge \mathrm{e} \in \mathrm{C}^{\mathcal{I}} \,\}$
nominal op.	$\{\mathrm{o}\}$	$\{\mathrm{o}^{\mathcal{I}}\}$
inverse role op.	p^{-}	$\{(\mathrm{d}, \mathrm{e}) \mid (\mathrm{e}, \mathrm{d}) \in p^{\mathcal{I}}\}$
role boolean op. for $\oplus \in \{\cup, \cap, \setminus\}$	$s_1 \oplus s_2$	$s_1^{\mathcal{I}} \oplus s_2^{\mathcal{I}}$
Self op.	$\exists s.\mathsf{Self}$	$\{\mathrm{d} \mid (\mathrm{d}, \mathrm{d}) \in s^{\mathcal{I}}\}$

Table 1: Concepts and roles in $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$.

Any triple $\mathcal{V} \triangleq (\sigma_i, \sigma_c, \sigma_r)$ from $\mathbf{N_I} \times \mathbf{N_C} \times \mathbf{N_R}$ having finite components will be called a *vocabulary*. We often speak about $\mathcal{L}(\mathcal{V})$-concepts *i.e.* those $\mathcal{L}$-concepts that employ only symbols from $\mathcal{V}$. For a *pointed interpretation* $(\mathcal{I}, \mathrm{d})$ we say that it *satisfies* a concept C (written: $(\mathcal{I}, \mathrm{d}) \models \mathrm{C}$) if $\mathrm{d} \in \mathrm{C}^{\mathcal{I}}$. An $\mathcal{V}$-pointed-interpretation $(\mathcal{I}, \mathrm{d})$ is a partial interpretation, where all indv-names outside $\mathcal{V}$ are left undefined while other other symbols outside $\mathcal{V}$ are interpreted as $\emptyset$.

2.2. Preliminaries on Category Theory

We assume familiarity with basic concepts such as categories, functors or natural transformations. Let $\mathbb{C}$ and $\mathbb{D}$ be categories. We write $|\mathbb{C}|$ to denote morphisms (arrows) of $\mathbb{C}$ and $f \in |\mathbb{C}|$ to indicate that f is a morphism in $\mathbb{C}$. Let $G : \mathbb{C} \to \mathbb{C}$ be a functor and $\varepsilon : \mathbb{C} \Rightarrow 1_{\mathbb{C}}$ a natural transformation, with $1_{\mathbb{C}}$ being the identity functor on $\mathbb{C}$. A *comonad* G is a triple $(G, \varepsilon, (\cdot)^*)$, where ε is called the *counit* of G that for each object A it gives us an arrow $\varepsilon_A : GA \to A$, while $(\cdot)^*$, called the *Kleisli coextension* of G, is an operator sending each arrow $f : GA \to B$ to $f^* : GA \to GB$. These has to satisfy, for all $f : GA \to B$ and $g : GB \to C$, the equations:

$$\varepsilon_A^* = 1_{GA}, \qquad \varepsilon_B \circ f^* = f, \qquad (g \circ f^*)^* = g^* \circ f^*$$

Furthermore, we define *coKleisli category* $\mathsf{Kl}(G)$, with objects from $\mathbb{C}$ and arrows from A to B given by the arrows in $\mathbb{C}$ of the form $GA \to B$, where composition $g \bullet f$ is given by $g \circ f^*$.

We shall also need the notion of *relative comonads* [12]. Given a functor $J : \mathbb{C} \to \mathbb{D}$, and a comonad G on $\mathbb{D}$, we obtain a *relative comonad* on $\mathbb{C}$, whose coKleisli category is defined as follows. A morphism from A to B, for objects A, B of $\mathbb{C}$, is a $\mathbb{D}$-arrow $GJA \to JB$. The counit at A is ε_{JA}, using the counit of G at JA. Given $f : GJA \to JB$, the Kleisli coextension $f^* : GJA \to GJB$ is the Kleisli coextension of G. Since G is a comonad, these operations satisfy the equations for a comonad in Kleisli form. We write this as $(G \circ J)$-relative-comonad.

2.3. Bisimulation Games

Let $\mathcal{V}$ be a vocabulary. Following [13], we recap the notion of *bisimulation games* for $\mathcal{ALC}$ and its extensions. Call $\mathrm{d} \in \Delta^{\mathcal{I}}$ and $\mathrm{e} \in \Delta^{\mathcal{J}}$ to be in $\mathcal{V}$*-harmony*[2] if for all concept names $\mathrm{C} \in \sigma_c$ we have that $\mathrm{d} \in \mathrm{C}^{\mathcal{I}}$ iff $\mathrm{e} \in \mathrm{C}^{\mathcal{J}}$. The $\mathcal{ALC}(\mathcal{V})$*-bisimulation game* is played by two players, Spoiler (he) and Duplicator (she), on two pointed interpretations $(\mathcal{I}, \mathrm{d}_0)$ and $(\mathcal{J}, \mathrm{e}_0)$. A *configuration* of a game is a quartet of the form $(\mathcal{I}, s; \mathcal{J}, s')$, where s and s' are words from, respectively, $\Delta^{\mathcal{I}}(\sigma_r \Delta^{\mathcal{I}})^*$ and $\Delta^{\mathcal{J}}(\sigma_r \Delta^{\mathcal{J}})^*$. Intuitively, configurations encode not only the current position of the play, but also its full play history. The *initial configuration* is simply $(\mathcal{I}, \mathrm{d}_0; \mathcal{J}, \mathrm{e}_0)$. The 0-th round of the game starts in the initial configuration and we require that d_0 and e_0 are in $\mathcal{V}$-harmony. If not, then immediately Spoiler wins. For any configuration $(\mathcal{I}, s\mathrm{d}; \mathcal{J}, s'\mathrm{e})$ (where the sequences s, s' may be empty) in the game, the following rules apply:

(a) In each round, Spoiler picks one of the two interpretations, say $\mathcal{I}$. Then he picks a role name $r \in \sigma_r$ and takes an element $\mathrm{d}' \in \Delta^{\mathcal{I}}$ such that $(\heartsuit)$: $(\mathrm{d}, \mathrm{d}') \in r^{\mathcal{I}}$. If there is no such role name r and an element d', then Duplicator wins.

(b) Duplicator responds in the other interpretation, $\mathcal{J}$, by picking the same role name $r \in \sigma_r$ as Spoiler did and an element $\mathrm{e}' \in \Delta^{\mathcal{I}}$ in $\mathcal{V}$-harmony with d', witnessing $(\clubsuit)$: $(\mathrm{e}, \mathrm{e}') \in r^{\mathcal{J}}$. If there is no such role name r or an element e', Spoiler wins.

The game continues from the position $(\mathcal{I}, s\mathrm{d}r\mathrm{d}'; \mathcal{J}, s'\mathrm{e}r\mathrm{e}')$. Duplicator has a winning strategy in the game on $(\mathcal{I}, \mathrm{d}_0; \mathcal{J}, \mathrm{e}_0)$ if she can respond to every move of Spoiler so that she either wins the game or can survive ω rounds. We define winning strategies in k-round games analogously.

The above game is adjusted to the case of expressive sublogics $\mathcal{L}\Phi$ of $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$ as follows.

- If $\mathcal{O} \in \Phi$, then we extend the definition of $\mathcal{V}$-harmony with a condition "for all $\mathrm{o} \in \sigma_i$ we have that $\mathrm{d} = \mathrm{o}^{\mathcal{I}}$ iff $\mathrm{e} = \mathrm{o}^{\mathcal{J}}$".
- If $\mathsf{Self} \in \Phi$, then we extend the definition of $\mathcal{V}$-harmony with a condition "for all $r \in \sigma_r$ we have that $(\mathrm{d}, \mathrm{d}) \in r^{\mathcal{I}}$ iff $(\mathrm{e}, \mathrm{e}) \in r^{\mathcal{J}}$".
- If $\mathcal{I} \in \Phi$, then in Spoiler's move the condition $(\heartsuit)$ additionally allows for $(\mathrm{d}', \mathrm{d}) \in r^{\mathcal{I}}$. Then in the corresponding move of Duplicator, the condition $(\clubsuit)$ imposes $(\mathrm{e}', \mathrm{e}) \in r^{\mathcal{J}}$.
- If $b \in \Phi$, then for the element e' we additionally extend $(\clubsuit)$ in order to fulfil the equality $\{r \in \sigma_r \mid (\mathrm{d}, \mathrm{d}') \in r^{\mathcal{I}}\} = \{r \in \sigma_r \mid (\mathrm{e}, \mathrm{e}') \in r^{\mathcal{J}}\}$. Moreover, in case of $\mathcal{I} \in \Phi$, then also $\{r \in \sigma_r \mid (\mathrm{d}', \mathrm{d}) \in r^{\mathcal{I}}\} = \{r \in \sigma_r \mid (\mathrm{e}', \mathrm{e}) \in r^{\mathcal{J}}\}$ must hold.

To simplify reasoning about bisimulation games, we employ the well-known notion of *bisimulation*, which can be seen as the "encoding" of winning strategies of Duplicator. Let $\mathcal{L}\Phi$ be an expressive sublogic of $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$ and $k \in \mathbb{N} \cup \{\omega\}$. Following [14], the $\mathcal{L}\Phi(\mathcal{V})$-$k$-*bisimulation* between $\mathcal{I}$ and $\mathcal{J}$ is a set $\mathcal{Z} \subseteq \bigcup_{\ell=0}^{k} (\Delta^{\mathcal{I}})^k \times (\Delta^{\mathcal{J}})^k$ satisfying the following seven conditions for all $\mathrm{o} \in \sigma_i, \mathrm{C} \in \sigma_c, r \in \sigma_r, \mathrm{d}, \mathrm{d}' \in \Delta^{\mathcal{I}}$, $s \in (\Delta^{\mathcal{I}})^*$ and $\mathrm{e}, \mathrm{e}' \in \Delta^{\mathcal{J}}$, $s' \in (\Delta^{\mathcal{J}})^*$:

[2] For $\mathcal{ALC}$ we do not actually use σ_i and σ_r, but they will be useful for other logics.

(a) If $\mathcal{Z}(s\mathrm{d}, s'\mathrm{e})$ then $\mathrm{d} \in \mathrm{C}^{\mathcal{I}}$ iff $\mathrm{e} \in \mathrm{C}^{\mathcal{J}}$.

(b) If $\mathcal{Z}(s\mathrm{d}, s'\mathrm{e})$ and $(\mathrm{d}, \mathrm{d}') \in r^{\mathcal{I}}$ then there is $\mathrm{e}' \in \Delta^{\mathcal{J}}$ s.t. $(\mathrm{e}, \mathrm{e}') \in r^{\mathcal{J}}$ and $\mathcal{Z}(s\mathrm{dd}', s'\mathrm{ee}')$.

(c) If $\mathcal{Z}(s\mathrm{d}, s'\mathrm{e})$ and $(\mathrm{e}, \mathrm{e}') \in r^{\mathcal{J}}$ then there is $\mathrm{d}' \in \Delta^{\mathcal{J}}$ s.t. $(\mathrm{d}, \mathrm{d}') \in r^{\mathcal{I}}$ and $\mathcal{Z}(s\mathrm{dd}', s'\mathrm{ee}')$.

(d) If $\mathcal{O} \in \Phi$, then $\mathcal{Z}(s\mathrm{d}, s'\mathrm{e})$ implies $\mathrm{d} = \mathrm{o}^{\mathcal{I}}$ iff $\mathrm{e} = \mathrm{o}^{\mathcal{J}}$.

(e) If $\mathsf{Self} \in \Phi$, then $\mathcal{Z}(s\mathrm{d}, s'\mathrm{e})$ implies $(\mathrm{d}, \mathrm{d}) \in r^{\mathcal{I}}$ iff $(\mathrm{e}, \mathrm{e}) \in r^{\mathcal{J}}$.

(f) If $\mathcal{I} \in \Phi$, then $\mathcal{Z}(s\mathrm{d}, s'\mathrm{e})$ and $(\mathrm{d}', \mathrm{d}) \in r^{\mathcal{I}}$ implies that there is $\mathrm{e}' \in \Delta^{\mathcal{J}}$ s.t. $(\mathrm{e}', \mathrm{e}) \in r^{\mathcal{J}}$ and $\mathcal{Z}(s\mathrm{dd}', s'\mathrm{ee}')$.

(g) If $b \in \Phi$, then if $\mathcal{Z}(s\mathrm{d}, s'\mathrm{e})$ and $(\mathrm{d}, \mathrm{d}') \in r^{\mathcal{I}}$ implies that there is $\mathrm{e}' \in \Delta^{\mathcal{J}}$ satisfying $\mathcal{Z}(s\mathrm{dd}', s'\mathrm{ee}')$ and $\{r \in \sigma_r \mid (\mathrm{d}, \mathrm{d}') \in r^{\mathcal{I}}\} = \{r \in \sigma_r \mid (\mathrm{e}, \mathrm{e}') \in r^{\mathcal{J}}\}$. If $\mathcal{I} \in \Phi$, then also $\{r \in \sigma_r \mid (\mathrm{d}', \mathrm{d}) \in r^{\mathcal{I}}\} = \{r \in \sigma_r \mid (\mathrm{e}', \mathrm{e}) \in r^{\mathcal{J}}\}$.

We write $(\mathcal{I}, \mathrm{d}) \equiv_k^{\mathcal{L}\Phi(\mathcal{V})} (\mathcal{J}, \mathrm{e})$ iff d and e satisfy the same $\mathcal{L}\Phi(\mathcal{V})$-concepts of rank at most k (as before, here k can be also ω). The following fact for most of considered logics is either well-known (see [13], in particular Prop. 2.1.3 and related chapters) or can be established by tiny modifications of existing proofs.

Fact 2.1. *For any $k \in \mathbb{N} \cup \{\omega\}$ and a logic $\mathcal{L}\Phi$ between $\mathcal{ALC}$ and $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$, t.f.a.e.:*

- *Duplicator has the winning strategy in the k-round $\mathcal{L}\Phi(\mathcal{V})$-bisim-game on $(\mathcal{I}, \mathrm{d}; \mathcal{J}, \mathrm{e})$,*
- *There is an $\mathcal{L}\Phi(\mathcal{V})$-$k$-bisimulation $\mathcal{Z}$ between $\mathcal{I}$ and $\mathcal{J}$ such that $\mathcal{Z}(\mathrm{d}, \mathrm{e})$,*
- $(\mathcal{I}, \mathrm{d}) \equiv_k^{\mathcal{L}\Phi(\mathcal{V})} (\mathcal{J}, \mathrm{e})$.

3. Reductions Between Games and Logics

Herein we establish reductions, based on appropriate model transformations, that will allow us for transferring winning strategies of Duplicator from richer logics to weaker ones, ending up on $\mathcal{ALC}$. All of them, except the case of nominals, will be trivial. Such transformation will be essential in Section 5, where we will employ them in the construction of relative comonads.

We will denote the game reductions for logic extensions Φ by $\mathfrak{f}_\Phi$, which has two components $\mathfrak{f}_\Phi^{\mathcal{I}}$ and $\mathfrak{f}_\Phi^*$, that define actions on, respectively, the interpretation and the distinguished element.

We first handle the Self operator. Let $\sigma_c^{\mathsf{Self}} \triangleq \sigma_c \cup \{\mathrm{C}_{\mathsf{Self}.r} \mid r \in \sigma_r\}$. By the *self-enrichment* of a $\mathcal{V} \triangleq (\sigma_i, \sigma_c, \sigma_r)$-interpretation $\mathcal{I}$ we mean the $\mathcal{V}^{\mathsf{Self}} \triangleq (\sigma_i, \sigma_c^{\mathsf{Self}}, \sigma_r)$-interpretation $\mathcal{I}_{\mathsf{Self}}$, where the $(\sigma_i, \sigma_c, \sigma_r)$-reduct[3] of $\mathcal{I}_{\mathsf{Self}}$ is equal to $\mathcal{I}$, and the interpretation of $\mathrm{C}_{\mathsf{Self}.r}$ concepts is defined as $(\mathrm{C}_{\mathsf{Self}.r})^{\mathcal{I}_{\mathsf{Self}}} = (\exists r.\mathsf{Self})^{\mathcal{I}}$. Let $\mathfrak{f}_{\mathsf{Self}}$ be the described transformation, mapping $\mathcal{I}$ to $\mathcal{I}_{\mathsf{Self}}$. The following proposition is immediate from the semantics of Self:

Proposition 3.1. *Let $k \in \mathbb{N} \cup \{\omega\}$ and let $\mathcal{L}$ be a DL satisfying $\mathcal{ALC} \subseteq \mathcal{L} \subseteq \mathcal{ALCI}b\mathcal{O}$. Then Duplicator has a winning strategy in a k-round $\mathcal{L}_{\mathsf{Self}}(\mathcal{V})$-bisimulation game on $(\mathcal{I}, \mathrm{d}; \mathcal{J}, \mathrm{e})$ iff she has a winning strategy in a k-round $\mathcal{L}(\mathcal{V})$-bisimulation game on $(\mathfrak{f}_{\mathsf{Self}}(\mathcal{I}), \mathrm{d}; \mathfrak{f}_{\mathsf{Self}}(\mathcal{J}), \mathrm{e})$.*

[3] i.e. the interpretation obtained from $\mathcal{I}_{\mathsf{Self}}$ by omitting the interpretation of symbols outside $\sigma_i \cup \sigma_c \cup \sigma_r$.

Proof. Employ Fact 2.1 after observing that due to the choice of σ_c^{Self} the $\mathcal{L}_{\mathsf{Self}}(\mathcal{V})$-$k$-bisimulation between $\mathcal{I}$ and $\mathcal{J}$ is a $\mathcal{L}(\mathcal{V}^{\mathsf{Self}})$-$k$-bisimulation between $\mathfrak{f}_{\mathsf{Self}}(\mathcal{I})$ and $\mathfrak{f}_{\mathsf{Self}}(\mathcal{J})$ and vice-versa. □

Our next goal is to incorporate inverses of roles. Let $\sigma_r^{\mathcal{I}} \triangleq \sigma_r \cup \{r_{inv} \mid r \in \sigma_r\}$ By the *inverse-enrichment* of a $\mathcal{V} \triangleq (\sigma_i, \sigma_c, \sigma_r)$-interpretation $\mathcal{I}$ we mean the $\mathcal{V}^{\mathcal{I}} \triangleq (\sigma_i, \sigma_c, \sigma_r^{\mathcal{I}})$-interpretation $\mathcal{I}_{\mathcal{I}}$, where the $(\sigma_i, \sigma_c, \emptyset)$-reducts of $\mathcal{I}$ and $\mathcal{I}_{\mathcal{I}}$ are equal, and the interpretations of role names r_{inv} are defined as $(r_{inv})^{\mathcal{I}_{\mathcal{I}}} = (r^-)^{\mathcal{I}}$. Let $\mathfrak{f}_{\mathcal{I}}$ be the described transformation, mapping $\mathcal{I}$ to $\mathcal{I}_{\mathcal{I}}$. The following follows analogously to Proposition 3.1:

Proposition 3.2. *Let $k \in \mathbb{N} \cup \{\omega\}$ and let $\mathcal{L}$ be a DL satisfying $\mathcal{ALC} \subseteq \mathcal{L} \subseteq \mathcal{ALCO}b$. Then Duplicator has a winning strategy in a k-round $\mathcal{LI}(\mathcal{V})$-bisimulation game on $(\mathcal{I}, \mathrm{d}; \mathcal{J}, \mathrm{e})$ iff she has a winning strategy in a k-round $\mathcal{L}(\mathcal{V}^{\mathcal{I}})$-bisimulation game on $(\mathfrak{f}_{\mathcal{I}}(\mathcal{I}), \mathrm{d}; \mathfrak{f}_{\mathcal{I}}(\mathcal{J}), \mathrm{e})$.*

We focus next on safe boolean combination of roles. Given a finite $\sigma_r \subseteq \mathbf{N_R}$, let σ_r^b be composed of role names having the form r_S, where S is any non empty subset of σ_r. By the *b-enrichment* of a $\mathcal{V} \triangleq (\sigma_i, \sigma_c, \sigma_r)$-interpretation $\mathcal{I}$ we mean the $\mathcal{V}^b \triangleq (\sigma_i, \sigma_c, \sigma_r^b)$-interpretation $\mathcal{I}_b$, where the $(\sigma_i, \sigma_c, \emptyset)$-reducts of $\mathcal{I}$ and $\mathcal{I}_b$ are equal and the interpretation of role names $r_S \in \sigma_r^b$ is defined as $\{(\mathrm{d}, \mathrm{e}) \mid \{r \in \sigma_r \mid (\mathrm{d}, \mathrm{e}) \in r^{\mathcal{I}}\} = S\}$. Let $\mathfrak{f}_b$ be the described transformation, mapping $\mathcal{I}$ to $\mathcal{I}_b$. Once more, the following proposition is straightforward:

Proposition 3.3. *Let $k \in \mathbb{N} \cup \{\omega\}$ and let $\mathcal{L}$ be a DL satisfying $\mathcal{ALC} \subseteq \mathcal{L} \subseteq \mathcal{ALCO}$. Then Duplicator has a winning strategy in a k-round $\mathcal{L}b(\mathcal{V})$-bisimulation-game on $(\mathcal{I}, \mathrm{d}; \mathcal{J}, \mathrm{e})$ iff she has a winning strategy in a k-round $\mathcal{L}(\mathcal{V}^b)$-bisimulation-game on $(\mathfrak{f}_b(\mathcal{I}), \mathrm{d}; \mathfrak{f}_b(\mathcal{J}), \mathrm{e})$.*

Finally, we proceed with the case of nominals. In this case we need to be extra careful, as the comonads introduces in the next section will act as unravelling on interpretations, and we do not want to create multiple copies of a nominal. Recall that the Gaifman graph $\mathsf{G}_{\mathcal{I}} = (V_{\mathcal{I}}, E_{\mathcal{I}})$ of an interpretation $\mathcal{I}$ is a simple undirected graph whose nodes are domain elements from $\Delta^{\mathcal{I}}$ and an edge exist between two nodes when there is a role that connects them in $\mathcal{I}$.

Let $\sigma_c^{\mathcal{O}} \triangleq \sigma_c \cup \{\mathrm{C}_{\mathrm{o},r} \mid \mathrm{o} \in \sigma_i, r \in \sigma_r\}$ and $\sigma_r^{\mathcal{O}} \triangleq \sigma_r \cup \{r_{\mathrm{o}} \mid \mathrm{o} \in \sigma_i\}$. By the *nominal-enrichment* of a $\mathcal{V} \triangleq (\sigma_i, \sigma_c, \sigma_r)$-interpretation $\mathcal{I}$ we mean the $\mathcal{V}^{\mathcal{O}} \triangleq (\sigma_i, \sigma_c^{\mathcal{O}}, \sigma_r^{\mathcal{O}})$-interpretation $\mathcal{I}_{\mathcal{O}}$ defined in the following steps:

- First, we get rid of unreachable elements from $\mathcal{I}$. More precisely, let $\mathcal{J}$ to be the substructure of $\mathcal{I}$ restricted to the set of all elements reachable in (finitely-many steps) from d in $\mathsf{G}_{\mathcal{I}}$. W.l.o.g., assume that all $\mathrm{o}^{\mathcal{I}}$ for $\mathrm{o} \in \sigma_i$ are reachable.
- For every $\mathrm{o} \in \sigma_i$ and every $\mathrm{d} \in \Delta^{\mathcal{I}}$ for which there is an r-connection between d and $\mathrm{o}^{\mathcal{I}}$, we insert a "trampoline" element labelled by the unique concept name $\mathrm{C}_{\mathrm{o},r}$ and we r-connect it with d. Trampoline elements are used to bookkeep information about connections between elements and named elements. Let $\mathcal{J}$ be the resulting interpretation.
- We next divide $\mathcal{J}$ into components. Let $\mathcal{J}_{\mathrm{d}}$ and $\mathcal{J}_{\mathrm{o}}$ for $\mathrm{o} \in \sigma_i$ be induced subinterpretations of $\mathcal{J}$ obtained by removing all elements $\{\mathrm{o}^{\mathcal{I}} \mid \mathrm{o} \in \sigma_i\}$ from $\mathcal{J}$ except the element mentioned in the subscript (that serve the role of distinguished elements of the components). Take $\mathcal{J}'$ to be the disjoint sum of the components.

- Finally, we will link components. For all $\mathrm{o} \in \sigma_i$, take dist_o be the length of the shortest path from d to $\mathrm{o}^{\mathcal{I}}$ in $\mathsf{G}_{\mathcal{I}}$. We will connect d to $\mathrm{o}^{\mathcal{J}'}$ by a dummy path of length precisely dist_o. Thus, we introduce dummy elements $\mathrm{d}_1^\mathrm{o}, \ldots, \mathrm{d}_{\mathrm{dist}_\mathrm{o}-1}^\mathrm{o}$ to $\Delta^{\mathcal{J}'}$ and employ the fresh role name r_o, whose interpretation will contains precisely the pairs $(\mathrm{d}, \mathrm{d}_1^\mathrm{o}), (\mathrm{d}_1^\mathrm{o}, \mathrm{d}_2^\mathrm{o}), \ldots, (\mathrm{d}_{\mathrm{dist}_\mathrm{o}-1}^\mathrm{o}, \mathrm{o}^{\mathcal{J}'})$. The resulting interpretation is the desired $\mathcal{I}_{\mathcal{O}}$.

Let $\mathfrak{f}_{\mathcal{O}}$ be the described transformation, mapping $\mathcal{I}$ to $\mathcal{I}_{\mathcal{O}}$. In Appendix we show that

Lemma 3.4. *Let $k \in \mathbb{N} \cup \{\omega\}$. Duplicator has a winning strategy in a k-round $\mathcal{ALCO}(\mathcal{V})$-bisimulation game on $(\mathcal{I}, \mathrm{d})$ and $(\mathcal{J}, \mathrm{e})$ iff she has a winning strategy in a k-round $\mathcal{ALC}(\mathcal{V}^{\mathcal{O}})$-bisimulation game on $(\mathfrak{f}_{\mathcal{O}}(\mathcal{I}), \mathrm{d})$ and $(\mathfrak{f}_{\mathcal{O}}(\mathcal{J}), \mathrm{e})$.*

We wrap up the above reductions, with a goal that the winning strategy of Duplicator in an $\mathcal{L}\Phi$-bisimulation game is equivalent to the winning strategy in a certain $\mathcal{ALC}$-bisimulation game. Note that the order of applications of reduction matters, *e.g.* we should apply first the $\mathfrak{f}_{\mathcal{I}}$ reduction, and only then $\mathfrak{f}_b$; otherwise we will not get all possible combinations of roles with inverse. Hence, we first proceed with $\mathfrak{f}_{\mathsf{Self}}$ reduction, then with $\mathfrak{f}_{\mathcal{I}}$, with $\mathfrak{f}_b$ and finally with $\mathfrak{f}_{\mathcal{O}}$. Let $\mathfrak{f}_\Phi$ be a composition of reductions for extensions $\Phi \in \{\mathsf{Self}, \mathcal{I}, b, \mathcal{O}\}$ in the above order.

Theorem 3.5. *Let $k \in \mathbb{N} \cup \{\omega\}$ and $\mathcal{L}\Phi$ satisfy $\mathcal{ALC} \subseteq \mathcal{L}\Phi \subseteq \mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$. Then Duplicator has a winning strategy in a k-round $\mathcal{L}\Phi(\mathcal{V})$-bisimulation game on $(\mathcal{I}, \mathrm{d})$ and $(\mathcal{J}, \mathrm{e})$ iff she has a winning strategy in a k-round $\mathcal{L}(\mathcal{V}^{\Phi})$-bisimulation game on $(\mathfrak{f}_\Phi(\mathcal{I}), \mathrm{d})$ and $(\mathfrak{f}_\Phi(\mathcal{J}), \mathrm{e})$.*

Proof. The key idea here is grounded on the composition of the reduction functions. Given Φ, we simply apply consecutively Propositions 3.1–3.3 and Lemma 3.4. □

4. Game Comonads

Having defined a family of game reductions, we are going to start employing basic category theory primitives to define denotational semantics for bisimulation games. In this section, we focus on vanilla $\mathcal{ALC}$. Since $\mathcal{ALC}$ is a notational variant of the multi-modal logic, it suffices to translate the work done in [2] to the description logic setting. Subsequently, we prove that such definition of a "generalised game" coincides with our definition of $\mathcal{ALC}(\mathcal{V})$-bisimulation game defined in Section 2.3. In what follows, we shall work in the category of pointed interpretations $\mathcal{R}_*(\mathcal{V})$ over a vocabulary $\mathcal{V}$, where objects $(\mathcal{I}, d)$ are $\mathcal{V}$-pointed-interpretations, and morphisms $h : (\mathcal{I}, d) \to (\mathcal{J}, e)$ are homomorphisms between interpretations that preserve the distinguished element. With $\mathbb{DL}_k^\Phi$, we will denote the corresponding game comonad, where k is the depth parameter and $\Phi \subseteq \{\mathsf{Self}, \mathcal{I}, b, \mathcal{O}\}$ parametrizes the set of language extensions. We will be a bit careless and write $\mathbb{DL}_k^{\mathcal{IO}}$ in place of $\mathbb{DL}_k^{\{\mathcal{I},\mathcal{O}\}}$, or likewise, $\mathbb{DL}_k$ to denote $\mathbb{DL}_k^{\{\}}$.

4.1. Comonad for $\mathcal{ALC}$

We start with introducing the comonad for $\mathcal{ALC}$, which will be the base for our further comonads.

Definition 4.1 ($\mathcal{ALC}$-comonad)**.** *For every $k \geq 0$, we define a comonad $\mathbb{DL}_k$ on $\mathcal{R}_*(\emptyset, \sigma_c, \sigma_r)$,*[4] *where $\mathbb{DL}_k$ unravels $(\mathcal{I}, \mathrm{d})$ from d, up to depth k.*[5] *More precisely:*

[4]Notice $\emptyset$ in place of σ_i. This is because $\mathcal{ALC}$-concepts cannot speak about individual names.
[5]For the notion of unravelling consult e.g. [11, Definition 3.21].

- *The domain of* $\mathbb{DL}_k(\mathcal{I}, d)$ *is composed of sequences* $[a_0, r_0, a_1, r_2, \ldots] \in \Delta^{\mathcal{I}}(\sigma_r \Delta^{\mathcal{I}})^*$, *where we additionally require that* $(a_i, a_{i+1}) \in r_i^{\mathcal{I}}$ *and* $a_0 = \mathrm{d}$. *The singleton sequence* $[\mathrm{d}]$ *serves as the distinguished element of* $\mathbb{DL}_k(\mathcal{I}, d)$.
- *The functorial action on morphisms for* $\mathbb{DL}_k$ *satisfies:*

$$\mathbb{DL}_k(h : (\mathcal{I}, d) \to (\mathcal{J}, e)) : \mathbb{DL}_k(\mathcal{I}, d) \to \mathbb{DL}_k(\mathcal{J}, e)$$
$$(\mathbb{DL}_k\, h)[a_0, \alpha_1, a_1, ..., \alpha_j, a_j] = [h\, a_0, \alpha_1, h\, a_1, ..., \alpha_j, h\, a_j]$$

- *The map* $\varepsilon_{\mathcal{I}} : \mathbb{DL}_k(\mathcal{I}, d) \to (\mathcal{I}, d)$ *sends a sequence to its last element.*
- *Concept names* $\mathrm{C} \in \sigma_c$ *are interpreted such that* $s \in \mathrm{C}^{\mathbb{DL}_k(\mathcal{I},d)}$ *iff* $\varepsilon_{\mathcal{I}} s \in \mathrm{C}^{\mathcal{I}}$.
- *For role names* $r \in \sigma_r$, *we put* $(s, t) \in r^{\mathbb{DL}_k(\mathcal{I},d)}$ *iff there is* $d' \in \Delta^{\mathcal{I}}$ *so that* $t = s[r, d']$.
- *For a morphism* $h : \mathbb{DL}_k(\mathcal{I}, d) \to (\mathcal{J}, e)$, *we define Kleisli coextension* $h^* : \mathbb{DL}_k(\mathcal{I}, d) \to \mathbb{DL}_k(\mathcal{J}, e)$ *recursively by* $h^*[d] = [e]$ *and* $h^*(s[\alpha, d']) = h^*(s)[\alpha, h(s[\alpha, d'])])$.

Proposition 4.2. *The triple* $(\mathbb{DL}_k, \varepsilon, (\cdot)^*)$ *is a comonad in Kleisli form on* $\mathcal{R}_*(\emptyset, \sigma_c, \sigma_r)$.

Having the $\mathcal{ALC}$-comonad defined, as the next step we introduce sufficient categorical background required to define bisimulation games in an abstract-enough way. This may be a bit heavy for readers not familiar enough with category theory.

4.2. Tree-like Structures, Paths and Embeddings

A *covering relation* $\prec$ for a partial order $\leq$ is a relation satisfying $x \prec y \triangleq x \leq y \wedge x \neq y \wedge (\forall z. x \leq z \leq y \implies z = x \vee z = y)$. This is employed to define tree-like structures below, that will intuitively serve as the description of bisimulation game strategies.

Definition 4.3. *A* ordered interpretation $(\mathcal{I}, d, \leq)$ *is a pointed interpretation* $(\mathcal{I}, \mathrm{d})$ *equipped with a partial order on* $\Delta^{\mathcal{I}}$ *such that* $\uparrow(d) \triangleq \{d' \in \Delta^{\mathcal{I}} \mid d \leq d'\}$ *is a tree order that satisfies the following condition* (***D***) *for* $x, y \in \uparrow(d)$, *we have* $x \prec y$ *iff* $(x, y) \in r^{\mathcal{I}}$ *for some* $r \in \sigma_r$. *Morphisms between ordered interpretations preserve the covering relation. We put* $\mathcal{R}^D_{*k}(\mathcal{V})$ *to be the category of ordered interpretation as objects with* k *bounding the height of the underlying tree.*

We next define different kinds of embeddings, essential to characterize plays.

Definition 4.4. *A morphism in* $\mathcal{R}^D_{*k}(\mathcal{V})$ *is an* embedding *if it is an injective strong homomorphism. We write* $\mathrm{e} : \mathcal{I} \rightarrowtail \mathcal{J}$ *to mean that* e *is an embedding. Now, we define a subcategory Paths of* $\mathcal{R}_*(\mathcal{V})$ *whose objects have tree orders that are linear, so they comprise a single branch. We say that* $\mathrm{e} : P \rightarrowtail \mathcal{I}$ *is a path embedding if* P *is a path. A morphism* $f : \mathcal{I} \to \mathcal{J} \in |\mathcal{R}^D_{*k}(\mathcal{V})|$ *is a pathwise embedding if for any path embedding* $\mathrm{e} : P \rightarrowtail \mathcal{I}$, $f \circ \mathrm{e}$ *is a path embedding.*

Let $\sqsubseteq$ being be the lexicographical order on sequences from $\Delta^{\mathcal{I}}$. From the construction of $\mathcal{R}^D_{*k}(\mathcal{V})$, we can extract a free functor, for which construction is justified in the Appendix:

Lemma 4.5. *There exists a canonical functor* $F_k\, \mathcal{I} = (\mathbb{DL}_k(\mathcal{I}, d), \sqsubseteq)$.

4.3. A Categorical View on Games

Given sufficient background, we can move on to the main result, namely, to the characterisation of $\equiv^{\mathcal{ALC}_k}$ in the language of category-theory. We start with defining what does it mean for a morphism in $f : \mathcal{I} \to \mathcal{J} \in |\mathcal{R}^D_{*k}(\mathcal{V})|$ to be *open*. This holds if, whenever we have a commutative square (LHS) then there is an embedding $Q \rightarrowtail \mathcal{I}$ such that the diagram on the RHS commutes.

$$\begin{tikzcd} P \arrow[r, tail] \arrow[d, tail] & Q \arrow[d, tail] \\ \mathcal{I} \arrow[r, "f"'] & \mathcal{J} \end{tikzcd} \qquad\qquad \begin{tikzcd} P \arrow[r, tail] \arrow[d, tail] & Q \arrow[d, tail] \arrow[dl, tail] \\ \mathcal{I} \arrow[r, "f"'] & \mathcal{J} \end{tikzcd}$$

Finally, we can define *back-and-forth equivalence* $(\mathcal{I}, d) \leftrightarrow_k^{\mathbb{DL}} (\mathcal{J}, e)$ between objects in $\mathcal{R}_*(\mathcal{V})$, intuitively corresponding to conditions (b) and (c) from the definition of a bisimulation. This holds if there is an object R in $\mathcal{R}^D_{*k}(\mathcal{V})$ and a span of open pathwise embeddings such that:

$$\begin{tikzcd} & R \arrow[dl, tail] \arrow[dr, tail] & \\ F_k(\mathcal{I}, d) & & F_k(\mathcal{J}, e) \end{tikzcd}$$

We shall now define a back-and-forth game $\mathcal{G}_k^\Phi(\mathcal{I}, d; \mathcal{J}, e)$ played between the interpretations $(\mathcal{I}, d)$ and $(\mathcal{J}, e)$. Positions of the game are pairs $(s, t) \in \mathbb{DL}_k^\Phi(\mathcal{I}, d) \times \mathbb{DL}_k^\Phi(\mathcal{J}, e)$. We define a relation $W(\mathcal{I}, d; \mathcal{J}, e) \subseteq \mathbb{DL}_k^\Phi(\mathcal{I}, d) \times \mathbb{DL}_k^\Phi(\mathcal{J}, e)$ as follows. A pair (s, t) is in $W(\mathcal{I}, d; \mathcal{J}, e)$ iff for some path P, *path embeddings* $e_1 : P \rightarrowtail \mathcal{I}$ and $e_2 : P \rightarrowtail \mathcal{J}$, and $p \in P$, $s = e_1\, p$ and $t = e_2\, p$. The intention is that $W(\mathcal{I}, d; \mathcal{J}, e)$ picks out the winning positions for Duplicator. At the start of each round of the game, the position is specified by $(s, t) \in \mathbb{DL}_k^\Phi(\mathcal{I}, d) \times \mathbb{DL}_k^\Phi(\mathcal{J}, e)$. The initial position is $([d], [e])$. The round proceeds as follows. Spoiler either chooses $s' \succ s$, and Duplicator must respond with $t' \succ t$, producing the new position (s', t'); or Spoiler chooses $t'' \succ t$, and Duplicator must respond with $s'' \succ s$, producing the new position (s'', t''). Duplicator wins the round if she is able to respond, and the new position is in $W(\mathcal{I}, d; \mathcal{J}, e)$. We follow the same notation convention as for $\mathbb{DL}_k^\Phi$ with respect to extensions Φ of game $\mathcal{G}_k^\Phi$. The following theorem follows from [2, Theorem 10.1].

Theorem 4.6. *Duplicator has a winning strategy in $\mathcal{G}_k(\mathcal{I}, d; \mathcal{J}, e)$ iff $(\mathcal{I}, d) \leftrightarrow_k^{\mathbb{DL}} (\mathcal{J}, e)$.*

The above theorem with aforementioned definitions were just slight variations of theorems and notions presented in [2]. We have accommodated them to the description logic setting and now we will glue them together with our definition of the bisimulation game from Section 2.3.

Theorem 4.7. *Given interpretations $(\mathcal{I}, d)$ and $(\mathcal{J}, e)$, the $\mathcal{G}_k(\mathcal{I}, d; \mathcal{J}, e)$ game for the $\mathbb{DL}_k$ comonad is equivalent to the k-round $\mathcal{ALC}(\mathcal{V})$-bisimulation game between $(\mathcal{I}, d)$ and $(\mathcal{J}, e)$.*

Proof. First note that configurations and the moves are structurally the same in both games. Hence, by induction over k it suffices to show that the winning conditions coincide.
Base. Let $k = 0$ and suppose $([d], [e]) \in W(\mathcal{I}, d; \mathcal{J}, e)$. That holds iff there are path embeddings $e_1 : P \rightarrowtail \mathcal{I}$, $e_2 : P \rightarrowtail \mathcal{J}$ and $p \in P$ such that $e_1\, p = [d]$ and $e_2\, p = [e]$. By strong

homomorphism property, d is in $\mathcal{V}$-harmony with p, which in turn is in $\mathcal{V}$-harmony with d, which by transitivity of $\mathcal{V}$-harmony concludes this case.

Step. Assume that the proposition holds for all $i \leq k$. We need to show that the winning conditions coincide for games of length $k+1$. Suppose $s = s'[\alpha_s, d'], t = t'[\alpha_t, e']$ and $(s,t) \in W(\mathcal{I}, d; \mathcal{J}, e)$. That holds iff there are path embeddings $e_1 : P \rightarrowtail \mathcal{I}$, $e_2 : P \rightarrowtail \mathcal{J}$ and $p \in P$ such that $e_1\, p = s$ and $e_2\, p = t$. By definition of $W(\mathcal{I}, d; \mathcal{J}, e)$ relation, we get that $(s', t') \in W(\mathcal{I}, d; \mathcal{J}, e)$ and hence, by induction hypothesis, s, t are a valid winning configuration in $\mathcal{ALC}$ game. It remains to show that $[\alpha_s, d']$ and $[\alpha_t, e']$ are valid moves leading to winning positions. From $e_1\, p = s$ and $e_2\, p = t$ we immediately get that $\alpha_s = \alpha_t$ and since e_1, e_2 are embeddings we have that d' is in $\mathcal{V}$-harmony with p which in turn is in $\mathcal{V}$-harmony with e', hence by transitivity of $\mathcal{V}$-harmony, we are done. $\square$

By applying Theorem 4.6, Theorem 4.7 and Fact 2.1, we derive our first result on comonadic semantics for description logic games, namely:

Theorem 4.8. *We have that* $(\mathcal{I}, d) \equiv^{\mathcal{ALC}_k} (\mathcal{J}, e)$ *if and only if* $(\mathcal{I}, d) \leftrightarrow_k^{\mathbb{DL}} (\mathcal{J}, e)$.

5. Comonads for Extensions of $\mathcal{ALC}$

We can now proceed with definitions of game comonads for extensions of $\mathcal{ALC}$.

The approach that we undertook relies on an observation that we had based on how I-morphisms were incorporated in [2]. In our case, relative comonads serve as a tool to start within the base category where our objects live, and then to enrich the interpretations encoding the additional capabilities available in bisimulation games for richer logics. We do this via the already-presented reductions from Section 3, followed by the notion of unravelling using $\mathbb{DL}_k$ defined in Section 4, all established in a generalised framework using relative comonads.

Definition 5.1. *A* vocabulary-map δ *is triple* $(\delta_i, \delta_c, \delta_r) : \mathbf{N_I} \times \mathbf{N_C} \times \mathbf{N_R} \to \mathbf{N_I} \times \mathbf{N_C} \times \mathbf{N_R}$ *that maps the vocabulary* $(\sigma_i, \sigma_c, \sigma_r) \longmapsto (\delta_i(\sigma_i), \delta_c(\sigma_c), \delta_r(\sigma_r))$.

Proposition 5.2. *Let* δ *be a vocabulary map and* $\mathfrak{f}$ *a game reduction. A* $(\mathfrak{f}, \delta)$-reduction-functor *is a full and faithful [9, Def 7.1] functor* $J : \mathcal{R}_*(\mathcal{V}) \to \mathcal{R}_*(\delta\, \mathcal{V})$ *acting* $(\mathcal{I}, d) \longmapsto (\mathfrak{f}^{\mathcal{I}}\, \mathcal{I}, \mathfrak{f}^*\, d)$.

While Proposition 5.2 is stated in a very general setting, we strictly only consider the reductions from Section 3. Knowing that, J is clearly a functor. The full and faithful property comes from the bidirectional nature of our reduction games, *i.e.* reductions are defined in a reversible way and as such the reduction-functor encodes a full subcategory. Thus, we obtain a family of $(\mathfrak{f}_\theta, \delta_\theta)$-reduction-functors, where $\theta \in \{\mathsf{Self}, \mathcal{I}, b, \mathcal{O}\}$ are considered logic extensions.

Definition 5.3. *Let* δ, δ' *be a vocabulary-maps. We say that a functor* $F : \mathcal{R}_*(\mathcal{V}) \to \mathcal{R}_*(\delta\, \mathcal{V})$ *is invariant over vocabulary-maps iff for any* δ' *it can be lifted to* $F : \mathcal{R}_*(\delta'\, \mathcal{V}) \to \mathcal{R}_*(\delta\, (\delta'\, \mathcal{V}))$

What want to capture by this that such a functor acting on $\mathcal{R}_*(\mathcal{V})$ category is natural in $\mathcal{V}$, *i.e.* does not depend on the contents of the concepts or roles. It is easy to see the following fact:

Observation 5.4. $\mathbb{DL}_k$ *and* $(\mathfrak{f}_\theta, \delta_\theta)$*-reduction-functors are invariant over vocabulary-maps.*

In order to obtain richer semantics, we shall leverage the functor composition, following the same order as defined for the game reductions in Section 3:

$$\mathcal{R}_*(\mathcal{V}) \xrightarrow{J_{\mathsf{Self}}} \mathcal{R}_*(\mathcal{V}^{\mathsf{Self}}) \xrightarrow{J_{\mathcal{I}}} \mathcal{R}_*(\mathcal{V}^{\mathsf{Self}\mathcal{I}}) \xrightarrow{J_b} \mathcal{R}_*(\mathcal{V}^{\mathsf{Self}\mathcal{I}b}) \xrightarrow{J_{\mathcal{O}}} \mathcal{R}_*(\mathcal{V}^{\mathsf{Self}\mathcal{I}b\mathcal{O}})$$

In Appendix we show that:

Lemma 5.5. *Reduction-functors are closed under composition.*

5.1. Comonadic Semantics for Extensions of $\mathcal{ALC}$

Having defined appropriate notions and tools, we now present the way to obtain game semantics for an arbitrary sublogic $\mathcal{ALC} \subseteq \mathcal{L}\Phi \subseteq \mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$ by the use of relative comonads.

Let $J_\Phi \triangleq \bigcirc_{\theta\in\Phi} J_\theta$ be a family of functors indexed by Φ where J_θ are $(\mathfrak{f}_\theta, \delta_\theta)$-reduction-functors and the operator $\bigcirc$ iterates over the extensions and composes the functors together in $(\mathsf{Self}, \mathcal{I}, b, \mathcal{O})$ order. It follows from Lemma 5.5 that J_Φ functors are also reduction-functors.

Proposition 5.6 ($\mathcal{ALC}\Phi$-comonad)**.** *The game comonad* $\mathbb{DL}_k^\Phi$ *is a* $(\mathbb{DL}_k \circ J_\Phi)$*-relative-comonad.*

Proof. We know that J_Φ is a functor. From Proposition 4.2 and Observation 5.4 we get that $\mathbb{DL}_k$ is a comonad on the codomain of J_Φ. Hence, by definition, $\mathbb{DL}_k^\Phi$ is a relative comonad. ☐

With that, we arrive at the concluding lemma which shall guide us to the final result.

Lemma 5.7. *Let $k \in \mathbb{N}\cup\{\omega\}$ and let $\Phi \subseteq \{\mathsf{Self}, \mathcal{I}, b, \mathcal{O}\}$. Given pointed interpretations $(\mathcal{I}, d)$ and $(\mathcal{J}, e)$, the $\mathcal{G}_k^\Phi(\mathcal{I}, d; \mathcal{J}, e)$ game for the $\mathbb{DL}_k^\Phi$ relative comonad is equivalent to the k-round $\mathcal{ALC}\Phi(\mathcal{V})$-bisimulation game played on $(\mathcal{I}, d)$ and $(\mathcal{J}, e)$.*

Proof. By Theorem 3.5, it suffices to show that $\mathcal{G}_k^\Phi(\mathcal{I}, d; \mathcal{J}, e)$ is equivalent to $\mathcal{ALC}(\mathcal{V}^\Phi)$-bisimulation game between $(\mathfrak{f}_\Phi^{\mathcal{I}}\ \mathcal{I}, \mathfrak{f}_\Phi^*\ d)$ and $(\mathfrak{f}_\Phi^{\mathcal{I}}\ \mathcal{J}, \mathfrak{f}_\Phi^*\ e)$. Recall that the positions in the $\mathcal{G}_k^\Phi(\mathcal{I}, d; \mathcal{J}, e)$ are pairs $(s,t) \in \mathbb{DL}_k^\Phi(\mathcal{I}, d) \times \mathbb{DL}_k^\Phi(\mathcal{J}, e)$. By unfolding the definition of $\mathbb{DL}_k^\Phi$, we get that it corresponds to a product of unravelings $(\mathfrak{f}_\Phi\ \mathcal{I}, d) \times (\mathfrak{f}_\Phi\ \mathcal{J}, e)$. Hence, s and t are sequences of the form $[a_0, \alpha_1, a_1, ..., \alpha_j, a_j]$, where $\alpha_i \in \sigma_r^\Phi$ and $a_i \in \Delta^{\mathcal{I}} \vee a_i \in \Delta^{\mathcal{J}}$ for $1 \leq i \leq j$. An attentive reader can already notice that it is the same as positions in the $\mathcal{ALC}(\mathcal{V})$-bisimulation game by definition in Section 2.3. What remains to be shown is that the winning conditions coincide. Note that after applying Theorem 3.5 we are playing the $\mathcal{ALC}$-bisimulation game, and thus the same inductive reasoning applies as in Theorem 4.7. ☐

We have finally arrived at the hearth of our result. This is summarised by the following theorem, which is an immediate corollary from Fact 2.1, Lemma 5.7 and Theorem 4.6.

Theorem 5.8. *For any $k \in \mathbb{N}\cup\{\omega\}$ and a logic $\mathcal{L}\Phi$ between $\mathcal{ALC}$ and $\mathcal{ALC}_{\mathsf{Self}}\mathcal{I}b\mathcal{O}$, t.f.a.e.:*

- *Duplicator has the winning strategy in the k-round $\mathcal{L}\Phi(\mathcal{V})$-bisim-game on $(\mathcal{I}, \mathrm{d}; \mathcal{J}, \mathrm{e})$,*
- *There is an $\mathcal{L}\Phi(\mathcal{V})$-$k$-bisimulation $\mathcal{Z}$ between $\mathcal{I}$ and $\mathcal{J}$ such that $\mathcal{Z}(\mathrm{d}, \mathrm{e})$,*
- $(\mathcal{I}, \mathrm{d}) \equiv_k^{\mathcal{L}\Phi(\mathcal{V})} (\mathcal{J}, \mathrm{e})$,
- $(\mathcal{I}, d) \leftrightarrow_k^{\mathbb{DL}^\Phi} (\mathcal{J}, e)$.

6. Conclusions

This paper provides yet another view on bisimulation games used in the description logic setting, via the lenses of comonadic semantics, as well as another nail for the comonads hammer developed in recent years. There are several directions for future work. One of them involves the analysis of other known DL extensions, e.g. the number restrictions or the universal role, where we believe that our approach can be useful. Another research direction is to investigate combinatorial properties arising from comonads [15, 2, 16], *e.g.* the coalgebra numbers, or deep dive into transcribing known theorems using the developed comonadic structure, as in [5, 6].

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